View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

Journal of Algebra 323 (2010) 3111–3123

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Some finiteness chain conditions on the set of intermediate rings

Ahmed Ayache

University of Bahrain, Faculty of Science, Department of Mathematics, P.O. Box: 32038, Suhkir, Bahrain

A R T I C L E   I N F O

Article history:
Received 5 November 2008
Available online 9 April 2010
Communicated by Michel Broué

Keywords:
Finite chain condition
Normal pair
Minimal extension
Finite maximal chain

A B S T R A C T

Let $R \subset S$ be an extension of integral domains and $[R, S]$ be the set of intermediate rings between R and S. We say that $[R, S]$ satisfies the finite chain condition (FCC) if every chain of distinct intermediate rings between R and S is finite. Our main purpose is to determine necessary and sufficient conditions so that $[R, S]$ satisfies FCC. We also investigate the relationship between FCC and other finiteness conditions. Several satisfactory results are settled, but special attention is focused on the case where R is a Prüfer ring or a Noetherian ring.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

All the rings are commutative integral domains.

Definition 1. Let $R \subset S$ be an extension of integral domains and denote by $[R, S]$ the set formed by the rings between R and S. We say that the extension $R \subset S$ (or the set $[R, S]$) satisfies *the finite chain condition* (*FCC*) *if each chain of distinct rings between* R and S is finite.

Clearly: $[R, S]$ satisfies FCC $\Longleftrightarrow$ $[R, S]$ satisfies ACC and DCC $\Longleftrightarrow$ each chain of distinct rings between R and S is finite.

Obviously, a maximal chain is formed by successive minimal extensions, that is by extensions with no proper intermediate ring.

Recalling that, when the extension $R \subset S$ is minimal, then either R is a field and then S is also a field (cf. [FO, Lemme 1.2]), or R is not a field and then R and S have the same quotient field (cf. [S, Preliminaries]), we easily obtain:

E-mail address: aayache@sci.uob.bh.

0021-8693/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2010.03.009

Proposition 2. *Let $R \subset S$ be an extension of integral domains.*

(i) *If R is a field, then $[R, S]$ satisfies FCC if and only if $R \subset S$ is an extension of fields with finite degree.*
(ii) *If R is not a field and $[R, S]$ satisfies FCC, then R and S have the same quotient field.*

So that, we may restrict our study to the following case:

$$R \subset S \subseteq qf(R)$$

where $qf(R)$ denotes the quotient field of R.

Gilmer characterized the case where $S = qf(R)$ (he called such a R, an *FC-domain*) by considering the integral closure $\overline{R}$ of R in $qf(R)$:

Proposition 3. *(See [G2, Theorem 2.14].) The extension $R \subset qf(R)$ satisfies FCC if and only if*

(i) *$\overline{R}$ is a Prüfer domain with finite spectrum.*
(ii) *$\overline{R}$ is a finite R-module.*
(iii) *R/C is an Artinian ring where $C = (R : \overline{R})$.*

The aim of this paper is to generalize Gilmer's result by replacing the quotient field $qf(R)$ of R by any overring S of R. We are led to introduce the ring $\overline{R}_S$, the integral closure of R in S. Analogously, we shall prove that the extension $R \subset S$ satisfies FCC if and only if both the extensions $R \subseteq \overline{R}_S$ and $\overline{R}_S \subseteq S$ satisfy FCC. We are led to study separately the case where R is integrally closed in S ($\overline{R}_S \subseteq S$, see Section 2) and S is integral over R ($\overline{R}_S = S$, see Section 3). Note that

- Gilmer's condition '$\overline{R}$ is a finite R-module' will naturally be replaced by '$\overline{R}_S$ is a finite R-module',
- '$R/(R : \overline{R})$ is an Artinian ring' will be replaced by '$(R : \overline{R}_S)$ is the intersection of finitely many maximal ideals of R',
- '$\overline{R}$ is a Prüfer domain with finite spectrum' will be replaced by 'every $T \in [\overline{R}_S, S]$ is integrally closed in S and $\{P \in Spec(R) \mid PS = S\}$ is finite'.

By the way, the property FCC for $[R, S]$ will be compared with both following properties:

- $[R, S]$ is finite (FI).
- $[R, S]$ contains a finite maximal chain from R to S (FMC).

We will establish the equivalences (FI) $\Longleftrightarrow$ (FCC) $\Longleftrightarrow$ (MFC) when R is integrally closed in S, and the equivalence (FCC) $\Longleftrightarrow$ (FMC) when $R \subset S$ is an integral extension. Moreover, some counterexamples will be built to show that the implications (FCC) $\Longrightarrow$ (FI) and (FMC) $\Longrightarrow$ (FCC) do not hold in general.

Finally, this study enables us to provide several interesting applications, precisely when R is a Prüfer ring or a Noetherian ring.

The proofs are based on the notion of *normal pair*s. A pair of rings (R, S) is said to be a *normal pair* provided that each $T \in [R, S]$ is integrally closed in S. These pairs where first defined and studied by E.D. Davis [D]. He proved that if R is local, then (R, S) is a normal pair if and only if there exists a divided prime ideal P of R (i.e, $PR_P = P$) such that $S = R_P$ and R/P is a valuation ring [D, Theorem 1]. Other characterizations of such pairs are also settled in [AJ]:

Proposition 4. *(See [AJ, Theorem 2.5 and Theorem 2.10].) If R is integrally closed in S, then the following assertions are equivalent*:

(i) *(R, S) is a normal pair.*
(ii) *For each $T \in [R, S]$, $Spec(T) = \{PT\colon\ PT \subset T,\ P \in Spec(R)\}$.*

(iii) *For each $T \in [R, S]$, the contraction $Spec(T) \to Spec(R)$ is injective.*
(iv) *For each $T \in [R, S]$, and $Q \in Spec(T)$; set $P = Q \cap R$, then $R_P = T_Q$.*
(v) *For each $T \in [R, S]$, $T = \bigcap_{P \in Spec(R), PT \subset T} R_P$.*

In particular, if R is local with maximal ideal M, the above assertions are equivalent to the following:

(vi) *Every $s \in S$ is a root of a polynomial which is not in $M[X]$.*
(vii) *For all $s \in S$, $s \in R$ or $s^{-1} \in R$.*

2. The case *R* is integrally closed in *S*

In this section, we will provide necessary and sufficient conditions under which $[R, S]$ satisfies FCC, when R is integrally closed in S. Define the *support* of R in S by

$$Supp(S/R) = \{P \in Spec(R)\colon\ PS = S\}.$$

This ordered set plays an important role in our study. We begin by some observations in the case where (R, S) is a normal pair.

Denote by $Max(R) = \{M_i\colon\ i \in I\}$ the set of all maximal ideals of R. For every maximal ideal M_i of R, the pair (R_{M_i}, S_{M_i}) is normal [D, Introduction]. Therefore, there is a prime ideal of R, say Q_i such that $S_{M_i} = R_{Q_i}$ and $Q_i R_{M_i}$ is a divided prime ideal of R_{M_i}. Let $\{Q_i\colon\ i \in I\}$ be the set of all prime ideals of R such that $S_{M_i} = R_{Q_i}$. It is easy to show that

$$Supp(S/R) = \{P \in Spec(R) \setminus \{0\}\colon\ P \nsubseteq Q_i,\ \forall i \in I\},$$

and that $Supp(S/R) = Spec(R) \setminus \{0\}$ if and only if S is the quotient field of R [AN, Proposition 1.1].

Recall the following result due to A. Jaballah [J, Lemma 3.2]. We label it as Lemma 5 for the sake of reference.

Lemma 5. *Suppose that R is integrally closed in S. Then $R \subset S$ is a minimal extension if and only if (R, S) is a normal pair and $|Supp(S/R)| = 1$.*

We will generalize this result to a finite maximal chain of intermediate rings. We begin by a formula concerning the cardinality of supports.

Lemma 6. *Let (R, S) be a normal pair. If $T \in [R, S]$, then $|Supp(T/R)| + |Supp(S/T)| = |Supp(S/R)|$.*

Proof. Let $Q \in Supp(S/R)$. Then either $QT = T$, so $Q \in Supp(T/R)$ or $QT \subset T$ (and $QS = S$), so $QT \in Spec(T)$ [Proposition 4]. Set

$$X = \{Q \in Spec(R)\colon\ QT \subset T,\ QS = S\}.$$

Then X and $Supp(T/R)$ form a partition of $Supp(S/R)$. Thus

$$|Supp(T/R)| + |X| = |Supp(S/R)|.$$

It remains to show that $|Supp(S/T)| = |X|$. Consider now the mapping $\phi : X \to Supp(S/T)$ which maps each element Q of X to QT. Then ϕ is onto since each prime ideal Q' of $Supp(S/T)$ can be written as $Q' = QT$ for some prime ideal Q of R [Proposition 4]. It is also injective since if Q_1 and Q_2 are two prime ideals of R such that $Q_1T = Q_2T \subset T$, then $Q_1 = Q_2$ by contraction on R [D, Proposition 4]. $\square$

Notice that, if the spectrum of R is finite, then the above result is clear since $Supp(S/R)$ is also finite and its cardinality is simply given by the following formula:

$$\left|Supp(S/R)\right| = \left|Spec(R)\right| - \left|Spec(S)\right|.$$

Proposition 7. *If R is integrally closed in S, then the following assertions are equivalent*:

(i) *There is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ in $[R, S]$.*
(ii) *(R, S) is a normal pair and $|Supp(S/R)| = n$.*

Proof. (i) $\Longrightarrow$ (ii) We start by proving that (R, S) is a normal pair. According to [D, Introduction], it suffices to show that (R_M, S_M) is a normal pair for every maximal ideal M of R. If M is a maximal ideal of R, then

$$R_M = (R_0)_M \subseteq (R_1)_M \subseteq \cdots \subseteq (R_n)_M = S_M$$

is a chain Δ between R_M and S_M such that $(R_i)_M = (R_{i+1})_M$ or $(R_i)_M \subset (R_{i+1})_M$ is a minimal extension. By refining the chain Δ, we obtain a finite maximal chain between R_M and S_M. Therefore, we may suppose that R is local with maximal ideal M. As R is integrally closed in R_1 and $R \subset R_1$ is a minimal extension, then (R, R_1) is clearly a normal pair. Thus $R_1 = R_Q$ for some prime ideal Q of R [D, Theorem 1]. Since R is integrally closed in R_2, then R_1 $(= R_Q)$ is integrally closed in $R_2 = (R_2)_Q$. It results that (R_1, R_2) is also a normal pair. Likewise, we can establish that (R_i, R_{i+1}) is a normal pair for each $0 \leqslant i \leqslant n-1$. Now, let P be a prime ideal of $S = R_n$, set $P_i = P \cap R_i$, then we have $P_i = P_{i+1} \cap R_i$ and $(R_{i+1})_{P_{i+1}} = (R_i)_{P_i}$ for each $0 \leqslant i \leqslant n-1$ [Proposition 4(iv)]. Progressively, it follows that $S_P = R_{P_0}$, and again by Proposition 4(iv), this ensures that (R, S) is a normal pair.

For the last assertion, it is sufficient to apply both Lemma 5 and Lemma 6 to get

$$\left|Supp(S/R)\right| = \sum_{i=0}^{n-1} \left|Supp(R_{i+1}/R_i)\right| = n.$$

(ii) $\Longrightarrow$ (i) Assume that (R, S) is a normal pair such that $|Supp(S/R)| = n$. We will argue by induction on n. If $n = 1$, then $R \subset S$ is a minimal extension, so Lemma 5 provides the answer. Suppose that this statement is true until $n-1$. There is necessarily a ring T such that $R \subset T \subset S$. Set $p = |Supp(T/R)|$ and $q = |Supp(S/T)|$, we have $n = |Supp(S/R)| = p + q$ with $p \leqslant n-1$ and $q \leqslant n-1$ [Lemma 6]. By induction theorem, there is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_p = T$ between R and T of length p and a finite maximal chain $T = R_p \subset R_{p+1} \subset \cdots \subset R_{p+q} = S$ between T and S of length q. Therefore, there is a finite maximal chain

$$R = R_0 \subset R_1 \subset \cdots \subset R_p \subset R_{p+1} \subset \cdots \subset R_{p+q} = R_n = S$$

between R and S of length n. $\square$

Corollary 8. *If R is integrally closed in S and $[R, S]$ satisfies FCC, then all (finite) maximal chains between R and S have the same length equal to $|Supp(S/R)|$.*

The following result determines necessary and sufficient conditions for $[R, S]$ to satisfy FCC and establishes the equivalence (FI) $\Longleftrightarrow$ (FCC) $\Longleftrightarrow$ (MFC) when R is integrally closed in S. In this situation, we find that (R, S) is a normal pair, $Supp(S/R)$ is finite and any intermediate ring in $[R, S]$ is a Kaplansky ideal transform. It is then convenient to recall the definition of such an ideal transform. For a nonzero ideal I of R, the *Kaplansky ideal transform* of R *with respect to* I is defined by

$$\Omega_R(I) = \left\{x \in qf(R)\colon\ \forall y \in I,\ xy^n \in R \text{ for some integer } n \geqslant 1\right\}.$$

$\Omega_R(I)$ is an overring of R and can be expressed in function of localizations of R as

$$\Omega_R(I) = \bigcap \{ R_P:\ P \in Spec(R),\ P \not\supseteq I \}.$$

Further properties of this ideal transform can be found in [F].

Theorem 9. *If R is integrally closed in S, then the following assertions are equivalent*:

(i) $[R, S]$ *satisfies FCC.*
(ii) *There exists a finite maximal chain from R to S.*
(iii) $[R, S]$ *is finite.*
(iv) (R, S) *is a normal pair and $Supp(S/R)$ is finite.*
(v) *d.c.c holds in $[R, S]$ and $Supp(S/R)$ is finite.*

Proof. (iii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (ii) results easily from a familiar argument.

(ii) $\Longrightarrow$ (iv) See Proposition 7.

(iv) $\Longrightarrow$ (iii) comes from [AN, Theorem 2.4], but we will develop here this implication for the sake of completeness. Consider the mapping Φ from the power set $P(Supp(S/R))$ to $[R, S]$ that maps $\emptyset$ to R, and any non-empty subset A of $Supp(S/R)$ to the Kaplansky ideal transform $\Omega_R(\prod_{Q \in A} Q)$. We claim that Φ is onto. Indeed, if $T \in [R, S]$, $T \neq R$, then T can be written as

$$T = \bigcap_{Q \in Spec(R),\ Q T \subset T} R_Q .$$

In the other way, note that $Supp(T/R)$ is finite since $Supp(T/R) \subseteq Supp(S/R)$. Therefore, one can verify directly that, for a prime ideal P of R, $P \notin Supp(T/R)$ if and only if $\prod_{Q \in Supp(T/R)} Q \nsubseteq P$. It follows that

$$T = \bigcap_{P \notin Supp(T/R)} R_P = \Omega_R\biggl(\prod_{Q \in Supp(T/R)} Q \biggr).$$

Therefore, Φ is onto and $|[R, S]| \leqslant 2^{|Supp(S/R)|} < \infty$.

(iii) $\Longrightarrow$ (v) d.c.c is clearly satisfied since $[R, S]$ is finite, and $Supp(S/R)$ is necessarily finite from (iv) (that is equivalent to (iii))

(v) $\Longrightarrow$ (iv) It remains to show that (R, S) is a normal pair. According to [D, Introduction], it suffices to show that (R_M, S_M) is a normal pair for every maximal ideal M of R. As d.c.c holds in $[R_M, S_M]$, we may suppose that R is local with maximal ideal M and proceed similarly to [G2, Proposition 1.1]: Let $s \in S$ and consider the decreasing chain of intermediate rings

$$R[s] \supset R[s^2] \supset R[s^4] \supset \cdots \supset R[s^{2^k}] \supset \cdots$$

in $[R, S]$. As $[R, S]$ satisfies d.c.c, then this chain stabilizes, so $s^{2^k} \in R[s^{2^{k+1}}]$ for some k. Thus s is a root of a polynomial $P(X)$ of $R[X]$ such that the coefficient of X^{2^k} in $P(X)$ is -1. Hence, $P(X) \notin M[X]$. Finally, as R is integrally closed in S, we conclude that (R, S) is a normal pair [Proposition 4]. $\square$

If R is a Prüfer ring and S is an overring of R, then (R, S) is a normal pair. We can derive the following result:

Corollary 10. *Let R be a Prüfer ring and S be an overring of R, then the following assertions are equivalent*:

(i) $[R, S]$ *satisfies FCC.*
(ii) $[R, S]$ *is finite.*
(iii) *Supp*(S/R) *is finite.*

In particular, if R has a finite spectrum, then $[R, S]$ is finite and all maximal chains between R and S have the same length equal to $|Spec(R)| - |Spec(S)|$.

3. The case *S* is integral over *R*

In this section, we will provide necessary and sufficient conditions under which $[R, S]$ satisfy FCC when S is integral over R. We start by the following useful result using conductors. Recall that if $R \subset S$ is an extension of rings, the *conductor* of S in R is defined by $(R : S) = \{x \in R\colon\ xS \subseteq R\}$, and that it is the largest ideal shared by R and S.

Lemma 11. *If $R \subset S$ is an integral extension of rings such that there is a maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ in $[R, S]$, then $(R : S)$ is the intersection of finitely many maximal ideals of R.*

Proof. According to [FO, Theorem 2.2], since $R_{i-1} \subset R_i$ is a minimal integral extension $(1 \leqslant i \leqslant n)$, there is a (unique) maximal ideal Q_i of R_{i-1} such that $Q_i R_i = Q_i$. Then $Q_i \subseteq (R_{i-1} : R_i)$, and so $Q_i = (R_{i-1} : R_i)$. It follows that $M_i = (R_{i-1} : R_i) \cap R$ is a maximal ideal of R. We claim that $(R : S) = \bigcap_{i=1}^n M_i$. Note first that $(R : S)$ is a common ideal of R_{i-1} and R_i, so $(R : S) \subseteq Q_i$ for each $1 \leqslant i \leqslant n$. Then $(R : S) \subseteq \bigcap_{i=1}^n Q_i = \bigcap_{i=1}^n M_i$. In the other way, as $Q_i R_i \subseteq R_{i-1}$ for all $1 \leqslant i \leqslant n$, then

$$\left(\prod_{i=1}^n Q_i\right) S \subseteq \left(\prod_{i=1}^{n-1} Q_i\right) R_{n-1} \subseteq \cdots \subseteq (Q_1 Q_2) R_2 \subseteq Q_1 R_1 \subseteq R.$$

It follows that $(\prod_{i=1}^n M_i) S \subseteq (\prod_{i=1}^n Q_i) S \subseteq R$, so $\prod_{i=1}^n M_i \subseteq (R : S)$. Thus $\sqrt{(R : S)} = \bigcap_{i=1}^n M_i$ and $M_1, M_2, \ldots, M_n$ are exactly the maximal ideals of R containing $(R : S)$. Now, we have $(R : S) = \bigcap_{M \in Max(R)} (R_M : S_M) \cap R$. As R and S share the nonzero ideal $(R : S)$, then $S_M = R_M$ for every maximal ideal M such that $(R : S) \nsubseteq M$. Thus $(R : S) = \bigcap_{i=1}^n (R_{M_i} : S_{M_i}) \cap R$. Finally, by proceeding in the same manner described in the proof of Proposition 7, we find that there is a finite maximal chain from R_{M_i} to S_{M_i}. In view of [G2, Lemma 2.6], the Jacobson ideal $J(R_{M_i}) = M_i R_{M_i}$ of R_{M_i} is a common ideal of R_{M_i} and S_{M_i}, and this implies that $M_i R_{M_i} = (R_{M_i} : S_{M_i})$. Hence $(R : S) = \bigcap_{i=1}^n (M_i R_{M_i}) \cap R = \bigcap_{i=1}^n M_i$. □

Theorem 12. *If $R \subset S$ is an integral extension of rings, set $C = (R : S)$, then the following assertions are equivalent*:

(i) $[R, S]$ *satisfies FCC.*
(ii) *There exists a finite maximal chain from R to S.*
(iii) *S is a finite R-module and C is the intersection of finitely many maximal ideals of R.*
(iv) *S/C has a finite length as R/C-module.*

Proof. (i) $\Longrightarrow$ (ii) comes by a straightforward manner. (ii) $\Longrightarrow$ (iii) Suppose that there is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$. By Lemma 11, we can say that C is the intersection of finitely many maximal ideals of R. Since $R \subset S$ is an integral extension, then to prove that S is a finite module over R, it suffices to show that S is finitely generated as a ring extension of R. Let $s_1 \in S \setminus R$. If $S = R[s_1]$, we are done, otherwise we take another element $s_2 \in S \setminus R[s_1]$. If $S = R[s_1, s_2]$, there is no more thing to prove, otherwise we pick an element $s_3 \in S \setminus R[s_1, s_2]$ and we repeat the same argument. As

$$R \subset R[s_1] \subset R[s_1, s_2] \subset R[s_1, s_2, s_3] \subset \cdots \subset S$$

is an increasing chain of intermediate rings between R and S, then this process must terminate since $[R, S]$ satisfies a.c.c. Hence $S = R[s_1, s_2, \ldots, s_n]$ for some elements $s_1, s_2, \ldots, s_n$ of S.

(iii) $\Longrightarrow$ (iv) S/C is a finite R/C-module. As C is the intersection of finitely many maximal ideals of R, then R/C is a finite direct sum of fields, so R/C is an Artinian ring. It follows that S/C has finite length as R/C-module.

(iv) $\Longrightarrow$ (i) Because S/C has finite length as R/C-module, the module S/C satisfies both the ascending and the descending chain conditions for R/C-submodules of S/C. This implies the validity of a.c.c and d.c.c for intermediate rings in $[R, S]$, so $[R, S]$ satisfies FCC, as was to be shown. □

We deduce the equivalence (FCC) $\Longleftrightarrow$ (FMC) in the case where $R \subset S$ is an integral extension. However, the implication (FCC) $\Longrightarrow$ (FI) is not true in general. For instance, let F be a field of characteristic $p \neq 0$, let X and Y be indeterminates over F, let $K = F(X, Y)$ and let $L = F(X^{\frac{1}{p}}, Y^{\frac{1}{p}})$. Then $K \subset L$ is a finite-dimensional field extension since $[L : K] = p^2$. On the other hand, because $L^P \subseteq K$, then $[K(u) : K] \leqslant p$ for each $u \in L$. It follows that L is not a simple extension of K. By application of the Primitive Element Theorem [AR, Theorem 26], there are infinitely many intermediate fields between K and L. If $S = L[\![T]\!]$ is the power series ring in an indeterminate T and $R = K + XL[\![T]\!]$, then $R \subset S$ is an integral extension. Furthermore, S is a finite R-module and $(R : S) = XL[\![T]\!]$ is the maximal ideal of R. Whence, $[R, S]$ satisfies FCC by Theorem 12, but $[R, S]$ has infinitely many intermediate rings.

As consequences of Theorem 12, we recover the following corollaries. We present here an analogue result to Proposition 7, but in the case where $R \subset S$ is an integral extension. For convenience, we will use the symbol $L_R(M)$ to denote the *length* of a module M over R.

Corollary 13. *If $R \subset S$ is an integral extension of rings, set $C = (R : S)$, then the following assertions are equivalent*:

(i) *There is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ in $[R, S]$.*
(ii) *C is the intersection of finitely many maximal ideals of R and $n = L_{R/C}(S/C) - L_{R/C}(R/C)$.*

Proof. (i) $\Longrightarrow$ (ii) According to Theorem 12, C is the intersection of finitely many maximal ideals of R and S/C is a finite R/C-module such that $L_{R/C}(S/C) < \infty$. As $R/C = R_0/C \subset R_1/C \subset \cdots \subset R_n/C = S/C$ is a finite maximal chain of R/C-modules between R/C and S/C, then

$$n = L_{R/C}\big((S/C)/(R/C)\big) = L_{R/C}(S/C) - L_{R/C}(R/C).$$

(ii) $\Longrightarrow$ (i) As C is the intersection of finitely many maximal ideals of R, then R/C is a finite direct sum of fields, so R/C is an Artinian ring and $L_{R/C}(R/C) < \infty$. It follows that $L_{R/C}(S/C) = L_{R/C}(R/C) + n < \infty$, and by application of Theorem 12, $[R, S]$ satisfies FCC. Therefore, there is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_m = S$ in $[R, S]$. But, from we have just proven before, we necessarily have $m = n = L_{R/C}(S/C) - L_{R/C}(R/C)$. □

The next corollaries continue in the same vein. They are fairly immediate of earlier results.

Corollary 14. *If $R \subset S$ is an integral extension and $[R, S]$ satisfies FCC, then all (finite) maximal chains between R and S have the same length equal to $L_{R/C}(S/C) - L_{R/C}(R/C)$.*

Corollary 15. *Let $R \subset S$ be an integral extension of rings such that R is local with maximal ideal m, then the following assertions are equivalent*:

(i) *$[R, S]$ satisfies FCC.*
(ii) *There is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ in $[R, S]$.*
(iii) *$m = (R : S)$ and S/m is an $(n+1)$-dimensional vector space over R/m.*

Let $R \subset S$ be an integral extension of rings such that $[R, S]$ satisfies FCC. If R is local and integrally closed in S, then (R, S) is a normal pair, so S is also local with maximal ideal $(R : S)$. However, if R is not integrally closed in S, then S may not be local.

Example 16. Let S be a semi-local Prüfer ring with quotient field $\mathbb{Q}(X)$ and n maximal ideals $\{M_i\colon 1 \leqslant i \leqslant n\}$ such that $S/M_i \cong \mathbb{Q}$ for each $i \in \{1, 2, \ldots, n\}$. It is easy to build such a ring as the intersection of n incomparable valuation overrings $V_i = \mathbb{Q}[X]_{(X+i)} = \mathbb{Q} + M_i$ of the polynomial ring $\mathbb{Q}[X]$. Set $I = \bigcap_{i=1}^{n} M_i$ and $R = \mathbb{Q} + I$, then R is local with maximal I. As $I \subseteq (R : S)$ and I is a maximal ideal of R, then $(R : S) = I$. Moreover, $S/I \cong \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ is a finite-dimensional vector space over $R/I \cong \mathbb{Q}$, so S is a finite R-module. By application of Corollary 15, we conclude that $[R, S]$ satisfies FCC.

4. The general case

A general characterization of $[R, S]$ which satisfies FCC merits further considerations. It is essentially based on the transitivity of the finite chain condition from $[R, \overline{R_S}]$ and $[\overline{R_S}, S]$ to $[R, S]$. Before embarking in this direction, we begin by some useful results concerning localizations.

Lemma 17. *If (R, S) is a normal pair of rings and N is a multiplicative set of R, then $|Supp(R_N, S_N)| \leqslant |Supp(R, S)|$.*

Proof. It is easy to verify that the function

$$\varphi : Supp(R_N, S_N) \to \{Q \in Supp(R, S),\ Q \cap N = \emptyset\}$$

which assigns to a prime ideal Q' of $Supp(R_N, S_N)$ its contraction $Q' \cap R$ on R is a bijective correspondence. $\square$

Remark 18. Let $R \subset S$ be an extension of rings. If $[R, S]$ satisfies FCC, then $[R_M, S_M]$ satisfies FCC for every maximal ideal M. But the converse is false. For instance, the pair $[\mathbb{Q}[X], \mathbb{Q}(X)]$ does not satisfy FCC since $[\mathbb{Q}[X], \mathbb{Q}(X)]$ is infinite [Theorem 9]. However, $[(\mathbb{Q}[X])_M, \mathbb{Q}(X)]$ satisfies FCC for every maximal ideal M of $\mathbb{Q}[X]$ since $(\mathbb{Q}[X])_M \subset \mathbb{Q}(X)$ is a minimal extension.

The following result provides a partial converse. Notice that the imposed condition (iii) holds when R is not integrally closed in S.

Proposition 19. *Let $R \subset S$ be an extension of rings. If*

(i) *$[R_M, S_M]$ satisfies FCC for every maximal ideal M of R.*
(ii) *$Supp(\overline{R_S}, S)$ is finite.*
(iii) *$(R : \overline{R_S})$ is the intersection of finitely many maximal ideals of R then $[R, S]$ satisfies FCC.*

Proof. By assumption, we have $C = (R : \overline{R_S}) = \bigcap_{i=1}^{n} M_i$, where $M_1, M_2, \ldots, M_n$ are maximal ideals of R. Let $R = R_0 \subseteq R_1 \subseteq \cdots$ be an ascending chain of intermediate rings between R and S. If M is a maximal ideal of R, then $R_M = (R_0)_M \subseteq (R_1)_M \subseteq \cdots$ is an ascending chain of intermediate rings between R_M and S_M.

- If $M \notin \{M_i\colon 1 \leqslant i \leqslant n\}$, then $R_M = (\overline{R_S})_M \subseteq S_M$. Since $[(\overline{R_S})_M, S_M]$ satisfies FCC, then $Supp((\overline{R_S})_M, S_M)$ is finite and any finite maximal chain between $(\overline{R_S})_M$ and S_M has length

$|Supp((\overline{R_S})_M, S_M)|$ [Corollary 8]. Therefore, by application of Lemma 17, for every $j \geqslant |Supp(\overline{R_S}, S)| \geqslant |Supp((\overline{R_S})_M, S_M)|$, we have $(R_j)_M = (R_{j+1})_M$.
– Now, if $M = M_i$ for some $i \in \{1, 2, \ldots, n\}$, we know by hypothesis that $[R_{M_i}, S_{M_i}]$ satisfies a.c.c, so there exists a positive integer k_i such that $(R_j)_{M_i} = (R_{j+1})_{M_i} = \cdots$ for every $j \geqslant k_i$.

Finally, for every $j \geqslant k = Max(|Supp(\overline{R_S}, S)|,\ k_1, k_2, \ldots, k_n)$, and for every maximal ideal M of R, we have $(R_j)_M = (R_{j+1})_M$. Thus, for every $j \geqslant k$, we have

$$R_j = \bigcap_{M \in Max(R)} (R_j)_M = \bigcap_{M \in Max(R)} (R_{j+1})_M = R_{j+1}.$$

Whence $[R, S]$ satisfies a.c.c. Similarly, one can prove the validity of d.c.c in $[R, S]$. Hence, $[R, S]$ satisfies FCC as desired. $\square$

Lemma 20. *Let $R \subset S$ be an integral extension of rings such that there is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ in $[R, S]$. If R is local with maximal ideal m, then S is semi-local with at most $n + 1$ maximal ideals.*

Proof. In light of Corollary 15, $m = (R : S)$ and S/m is an $(n + 1)$-dimensional vector space over R/m. If S has more than $n + 1$ maximal ideals, we can always consider r maximal ideals of S, namely $M_1, M_2, \ldots, M_r$, where $r > n + 1$, and build an increasing chain

$$m \subseteq M_1 M_2 \cdots M_r \subset M_1 M_2 \cdots M_{r-1} \subset \cdots \subset M_1 M_2 \subset M_1 \subset S,$$

of ideals of S. But this leads to an increasing chain

$$(0) \subseteq (M_1 M_2 \cdots M_r)/m \subset (M_1 M_2 \cdots M_{r-1})/m \subset \cdots \subset M_1/m \subset S/m$$

of R/m-subspaces of the vector space S/m with at least r terms, a contradiction since S/m is $(n + 1)$-dimensional. $\square$

Recall a needed definition [AJ]: Let $R \subset S$ be an extension of rings. We say that $R \subset S$ is a *residually algebraic* if, for each prime ideal Q of S and $P = Q \cap R$, the extension $R/P \subset S/Q$ is an algebraic extension. The pair of rings (R, S) is said to be *residually algebraic* if the extension $R \subset T$ is residually algebraic for each intermediate ring $T \in [R, S]$.

Residually algebraic pairs are strongly related to normal pairs. [AJ, Remark 2.2 and Theorem 2.10] revealed the relationship between them:

$$(R, S) \quad \text{is residually algebraic pair} \quad \Longleftrightarrow \quad (\overline{R_S} S) \quad \text{is a normal pair.}$$

Lemma 21. *Let (R, S) be a residually algebraic pair of rings. If E and F are two intermediate rings of $[R, S]$ such that*

(i) $E \subseteq F$.
(ii) $Max(E) = \{Q \cap E\colon\ Q \in Max(F)\}$.
(iii) $E \cap \overline{R_S} = F \cap \overline{R_S}$.

Then $E = F$.

Proof. This result is clear if $\overline{R_S} = R$. Let us suppose that $R \subset \overline{R_S}$. Set $T = E \cap \overline{R_S} = F \cap \overline{R_S}$. We claim that (T, F) is a normal pair, indeed, if $z \in F$ is integral over T, then z is integral over R, so $z \in F \cap \overline{R_S} = T$. Thus T is integrally closed in F. As (T, F) is a residually algebraic pair, then (T, F)

is a normal pair. It follows that $F_Q = E_{Q\cap E}$ for every prime ideal Q of F [Proposition 4], and in consequence, we obtain

$$F = \bigcap_{Q\in Max(F)} F_Q = \bigcap_{Q\in Max(F)} E_{Q\cap E} = E. \qquad \square$$

Proposition 22. *Let $R \subset S$ be an extension of rings. Then $[R,S]$ satisfies FCC if and only if $[R,\overline{R_S}]$ and $[\overline{R_S},S]$ satisfy FCC.*

Proof. It is clear that if $[R,S]$ satisfies FCC, then $[R,\overline{R_S}]$ and $[\overline{R_S},S]$ satisfy FCC. Let us prove the converse. Note that there is nothing to prove if $\overline{R_S} = R$ or $\overline{R_S} = S$, so we will suppose that $R \subset \overline{R_S} \subset S$. Since $[\overline{R_S},S]$ satisfies FCC, then $(\overline{R_S},S)$ is a normal pair [Theorem 9], so (R,S) is a residually algebraic pair. In light of Proposition 19, we may suppose that R is local with maximal ideal M. Assume that both chain conditions a.c.c and d.c.c hold in $[R,\overline{R_S}]$ and $[\overline{R_S},S]$, and let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending chain of intermediate rings in $[R,S]$. Using bars to denote integral closure in S, then $\overline{E_1} \subseteq \overline{E_2} \subseteq \cdots$ is an ascending chain of intermediate rings in $[\overline{R_S},S]$, so $\overline{E_p} = \overline{E_{p+1}} = \cdots$ for some p since $[\overline{R_S},S]$ satisfies a.c.c. It follows that each E_{p+i} is integral over E_p and each maximal ideal of E_{p+i} is the contraction of a maximal ideal of E_{p+i+1} for each i. Therefore, we have

$$\left|Max(E_p)\right| \leqslant \left|Max(E_{p+1})\right| \leqslant \cdots \leqslant \left|Max(\overline{R_S})\right|,$$

with the last inequality holding because, by Lemma 20, $\overline{R_S}$ is semi-local and $\overline{E_p}$ is an overring of $\overline{R_S}$ [AJ, Theorem 3.10]. Thus, there exists $q \geqslant p$ such that

$$\left|Max(E_q)\right| = \left|Max(E_{q+1})\right| = \cdots = n.$$

That means $Max(E_i) = \{M_{i,j}\colon\ 1 \leqslant j \leqslant n\}$ and $M_{i+1,j}$ lies over $M_{i,j}$ in E_i for every $i \geqslant q$ and $1 \leqslant j \leqslant n$. Thus $Max(E_i) = \{M_{i+1,j} \cap E_i\colon\ 1 \leqslant j \leqslant n\}$ for every $i \geqslant q$. In the other way, $E_1 \cap \overline{R_S} \subseteq E_2 \cap \overline{R_S} \subseteq \cdots$ is an ascending chain of intermediate rings in $[R,\overline{R_S}]$. Because a.c.c holds in $[R,\overline{R_S}]$, there exists $r \geqslant q$ such that $E_r \cap \overline{R_S} = E_{r+1} \cap \overline{R_S} = \cdots$. Finally, by using Lemma 21, we conclude that $E_r = E_{r+1} = \cdots$. Whence, a.c.c holds in $[R,S]$.

A similar argument shows that d.c.c also holds in $[R,S]$. We abbreviate here details, let $E_1 \supseteq E_2 \supseteq \cdots$ be a decreasing chain of intermediate rings in $[R,S]$. Then $\overline{E_1} \supseteq \overline{E_2} \supseteq \cdots$ is a decreasing chain of intermediate rings in $[\overline{R_S},S]$, so $\overline{E_p} = \overline{E_{p+1}} = \cdots$ for some p since $[\overline{R_S},S]$ satisfies d.c.c. Thus, there exists $q \geqslant p$ such that

$$\left|Max(E_q)\right| = \left|Max(E_{q+1})\right| = \cdots = n,$$

and $Max(E_{i+1}) = \{M_{i,j} \cap E_{i+1}\colon\ 1 \leqslant j \leqslant n\}$ for every $i \geqslant q$. In the other way, $E_1 \cap \overline{R_S} \supseteq E_2 \cap \overline{R_S} \supseteq \cdots$ is a decreasing chain of intermediate rings in $[R,\overline{R_S}]$. Because d.c.c holds in $[R,\overline{R_S}]$, there exists $r \geqslant q$ such that $E_r \cap \overline{R_S} = E_{r+1} \cap \overline{R_S} = \cdots$. Again, by using Lemma 21, we conclude that $E_r = E_{r+1} = \cdots$. $\square$

We are now in position to give our principal characterization of $[R,S]$ which satisfies FCC. Indeed, by combining Theorem 9 and Theorem 12 with Proposition 22, we derive directly:

Theorem 23. *Let $R \subset S$ be an extension of rings. Then $[R,S]$ satisfies FCC if and only if*

(i) *$(\overline{R_S},S)$ is a normal pair and $Supp(S/\overline{R_S})$ is finite.*
(ii) *$\overline{R_S}$ is a finite R-module.*
(iii) *$(R:\overline{R_S})$ is the intersection of finitely many maximal ideals of R.*

Again, by combining Theorem 9 and Theorem 12 with Proposition 22, we obtain another characterization of $[R,S]$ which satisfies FCC, in term of finite maximal chains.

Theorem 24. *Let $R \subset S$ be an extension of rings. Then $[R,S]$ satisfies FCC if and only if there is a finite maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_n = S$ in $[R,S]$ such that one of the R_i is $\overline{R_S}$.*

The following example shows that, if $\overline{R_S}$ does not belong to $\{R_0, R_1, \ldots, R_n\}$, then Theorem 24 does not follow. Furthermore, it shows that (FMC) does not imply (FCC) in general.

Example 25. This example is due to Kaplansky [K, Theorem 100]. Let k be a field of characteristic 2, and $T = k[\![X]\!]$, the power series ring in an indeterminate X. Let $u \in T$, $K = k(X, u^2)$ and $L = k(X, u)$. Then $D = T \cap K$ and $\overline{D}_L = T \cap L$ are discrete valuation rings with quotient fields K and L respectively. Moreover, $\overline{D}_L$ is the integral closure of D in L. It is possible to arrange $[L:K] = 2$, by taking X and u to be algebraically independent over k, in this situation, $\overline{D}_L$ is not a D-finite module. Let Y be an indeterminate over L, $S = L[Y]$, $H = K + YL[Y]$ and $R = D + YL[Y]$. As $D \subset K \subset L$ is a finite maximal chain, then $R \subset H \subset S$ is a finite maximal chain in $[R,S]$. In the other way, note that $\overline{R_S} = \overline{D}_L + YL[Y]$ is the integral closure of R in S. As $\overline{D}_L$ is not a finite D-module, then $\overline{R_S}$ is not a R-finitely generated module. It follows that $[R,S]$ does not satisfies FCC [Theorem 23].

If $S = K$ is a field and $[R,K]$ satisfies FCC, then K is the quotient field of R. Therefore $Supp(K/R)$ is exactly $Spec(R) - \{0\}$, and $\overline{R}$ is a Prüfer ring [AJ, Corollary 2.8]. By application of Theorem 23, we get a characterization of $[R,S]$ which satisfies FCC, when the second coordinate S is a field.

Corollary 26. *Let $R \subset K$ be an extension of rings, where K is a field. Then $[R,K]$ satisfies FCC if and only if*

(i) *K is the quotient field of R.*
(ii) *$\overline{R}$ is a Prüfer ring with finite spectrum.*
(iii) *$(R:\overline{R})$ is the intersection of some maximal ideals of R.*
(iv) *$\overline{R}$ is a finite R-module.*

Notice that [G2, Theorem 2.7] has already given a characterization of FC-domains similar to Corollary 26, but the condition (iii′) is stated instead of the condition (iii), where

(iii′) The Jacobson ideal $J(R)$ of R is an ideal of $\overline{R}$.

In fact, if $\overline{R}$ has a finite maximal spectrum, then (iii) and (iii′) are equivalent. The following lemma explains this fact.

Lemma 27. *Let R be a semi-local domain and S be a proper overring of R. Then $J(R)$ is an ideal of S if and only if $(R:S)$ is the intersection of some maximal ideals of R.*

Proof. It is clear that, if $(R:S)$ is the intersection of some maximal ideals of R, then $J(R) \subseteq (R:S)$, so $J(R)$ is a common ideal of R and S. Conversely, suppose that $J(R)$ is an ideal of S and let $Max(R) = \{M_1, M_2, \ldots, M_n\}$ be the set of maximal ideals of R. We have

$$(R:S) = \bigcap_{i=1}^{n} (R_{M_i} : S_{M_i}) \cap R.$$

Set $N = R - M_j$ for a fixed $j \in \{1, 2, \ldots, n\}$, then we have the obvious inclusions:

$$N^{-1}J(R) = \bigcap_{i=1}^{n} N^{-1}M_i = M_j R_{M_j} \subseteq N^{-1}(R:S) \subseteq (R:S)R_{M_j} \subseteq (R_{M_j} : S_{M_j}).$$

Thus $(R_{M_j} : S_{M_j}) = R_{M_j}$ or $(R_{M_j} : S_{M_j}) = M_j R_{M_j}$. If J is the set of all $1 \leqslant j \leqslant n$ for which $(R_{M_j} : S_{M_j}) = M_j R_{M_j}$, then $J \neq \emptyset$. Indeed, if J is empty, we get

$$(R : S) = \bigcap_{i=1}^{n} (R_{M_i} : S_{M_i}) \cap R = \bigcap_{i=1}^{n} R_{M_i} \cap R = R,$$

but this leads to the contradiction $R = S$. Hence

$$(R : S) = \bigcap_{i=1}^{n} (R_{M_i} : S_{M_i}) \cap R = \bigcap_{i \in J} M_i R_{M_i} \cap R = \bigcap_{i \in J} M_i. \qquad \square$$

Pullback construction is a useful tool for providing examples and counter-examples. The following result explores the finite chain condition on pullback rings.

Corollary 28. *Let S be an integral domain, M a maximal ideal of S, D a subring of the residue field $L = S/M$ and $R = \varphi^{-1}(D)$ the inverse image of D by the canonical epimorphism $\varphi : S \to L$.*

(1) *If D is a field, then $[R, S]$ satisfies FCC if and only if $[D : L] < \infty$.*
(2) *If D is not a field, then $[R, S]$ satisfies FCC if and only if*
   (i) *L is the quotient field of D.*
   (ii) *$\overline{D}$ is a finite D-module.*
   (iii) *$(D : \overline{D})$ is the intersection of maximal ideals of D.*
   (iv) *$\overline{D}$ is a Prüfer ring with finite spectrum.*

The next application treats the case where R is Noetherian.

Proposition 29. *Let R be a Noetherian ring and $R \subset S$ be an extension of rings. Then $[R, S]$ satisfies FCC if and only if*

(i) *$\overline{R_S}$ has a finite number of height-one maximal ideals M such $MS = S$.*
(ii) *$(R : \overline{R_S})$ is the intersection of finitely many maximal ideals of R.*
(iii) *$(\overline{R_S}, S)$ is a normal pair.*

Proof. Note first that, if $(R : \overline{R_S}) \neq 0$, then pick a nonzero element $x \in (R : \overline{R_S})$, we get $\overline{R_S} \subseteq \frac{1}{x}R$. As R is Noetherian, then $\overline{R_S}$ is a finite R-module and $\overline{R_S}$ is Noetherian. Therefore, according to Theorem 23, to prove the requested equivalence, it remains to show that $Supp(S/\overline{R_S})$ is exactly the set of maximal ideals M of $\overline{R_S}$ such that $ht_{\overline{R_S}}(M) = 1$ and $MS = S$. But this follows easily if we can show that every prime ideal Q of $Supp(S/\overline{R_S})$ is of height 1. Let $Q \in Supp(S/\overline{R_S})$. The extension $((\overline{R_S})_Q, S_Q)$ is a normal pair, so it is a Noetherian pair [AJ, Proposition 4.7]. If $ht_{\overline{R_S}}(Q) \geqslant 2$, then $(\overline{R_S})_Q = S_Q$ [W, Theorem 9]. Hence, $Q = Q(\overline{R_S})_Q \cap \overline{R_S} = (QS)S_Q \cap \overline{R_S}$, a contradiction since by assumption $QS = S$. $\square$

If R is Noetherian, then $\overline{R}$ is a Krull domain [N, Theorem 33.10]. If, in addition, R is an FC-domain, then $\overline{R}$ is a Prüfer domain, so $\overline{R}$ is a Dedekind domain [G1, Theorem 43.16]. We can then deduce, a characterization of FC-domains in the Noetherian case:

Corollary 30. *A Noetherian ring R is an FC-domain if and only if*

(i) *$(R : \overline{R})$ is the intersection of maximal ideals of R.*
(ii) *$\overline{R}$ is a semi-local principal ideal domain.*

Acknowledgment

The author thanks the referee for his several helpful remarks concerning the final form of this paper.

References

[AR] E. Artin, Galois Theory, second ed., Notre Dame Math. Lectures, vol. 2, Univ. Notre Dame, 1955.
[AJ] A. Ayache, A. Jaballah, Residually algebraic pairs of rings, Math. Z. 225 (1997) 49–65.
[AN] A. Ayache, N. Jarboui, Intermediary rings in a normal pair, J. Pure Appl. Algebra 212 (2008) 2176–2181.
[D] E.D. Davis, Overring of commutative rings III: Normal pairs, Trans. Amer. Math. Soc. 182 (1973) 175–185.
[F] M. Fontana, Kaplansky ideal transform: A survey, in: Advances in Commutative Ring Theory, Proc. of Third Fès (Morocco) Conference on Commutative Algebra, in: Lect. Notes Pure Appl. Math., vol. 205, Dekker, New York, 1999, p. 51.
[FO] D. Ferrand, J.P. Olivier, Homomorphisms minimaux d'anneaux, J. Algebra 16 (1970) 461–471.
[G1] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
[G2] R. Gilmer, Some finiteness conditions on the set of overrings of an integral domains, Proc. Amer. Math. Soc. 131 (8) (2003) 2337–2346.
[J] A. Jaballah, A Lower bound for the number of intermediary rings, Comm. Algebra 27 (3) (1999) 1307–1311.
[K] I. Kaplansky, Commutative Rings, revised ed., University of Chicago Press, Chicago, 1974.
[N] M. Nagata, Local Rings, Interscience, New York, 1962.
[S] J. Sato, T. Sugatini, K.I. Yochida, On minimal overring of a Noetherian domain, Comm. Algebra 20 (6) (1992) 1746–1753.
[W] A.R. Wadsworth, Pairs of domains where all intermediate domains are Noetherian, Trans. Amer. Math. Soc. 195 (1974) 201–211.